

ENTROPY AND PERIODIC ORBITS FOR EQUIVALENT SMOOTH FLOWS

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ABSTRACT. Given any $K > 0$, we construct two equivalent C^2 flows, one of which has positive topological entropy larger than K and admits zero as the exponential growth of periodic orbits, in contrast, the other has zero topological entropy and super-exponential growth of periodic orbits. Moreover we establish a C^∞ flow on \mathbb{S}^2 with super-exponential growth of periodic orbits, which is also equivalent to another flow with zero exponential growth of periodic orbits. On the other hand, any two dimensional flow has only zero topological entropy.

1. INTRODUCTION

It is a major goal in the theory of dynamical systems to determine the mechanisms which create deterministic chaos. One key ingredient causing chaotic behavior is the positivity of entropy, and the larger the entropy is, the more complicated the dynamics. It is of interesting to calculate the entropy for particular maps under study, but any of the standard definitions of entropy makes this a difficult task. A classical way for proving that a map f has positive topological entropy is to show that the number of periodic orbits of period n for f grows exponentially fast when $n \rightarrow \infty$. This motivation came from many analysis and summary on dynamical structures. In his prize essay [33], H. Poincaré was the first to imagine around 1890 the existence of transverse homoclinic intersections, that later was proved to be the limit of infinitely many periodic points by G. D. Birkhoff [5]. In 1965, S. Smale introduced a general geometrical model: Horseshoe contains all the complicated phenomena discovered by Poincaré and Birkhoff, and can also be described by symbolic coding. Indeed, homoclinic intersections give birth to very rich dynamics: positive topological entropy and infinity periodic points. Actually, the coexistence of the positive entropy and the positive exponential growth of periodic points has been established for open dense systems [34, 7, 15]. So, in most situations, “positive topological entropy” is synonymous of “many periodic orbits”.

Let M be a compact Riemannian manifold without boundary. Denote by $\text{Diff}^r(M)$ the set of C^r diffeomorphisms of M and, by $\mathcal{X}^r(M)$ the set of C^r vector fields on M , both endowed with the C^r topology, respectively.

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For $f \in \text{Diff}^r(M)$, denote the set of isolated periodic points of period n (i.e. the isolated fixed points of f^n) by

$$P_n(f) = \{ \text{isolated } x \in M \mid f^n(x) = x \},$$

and define the exponential growth rate of periodic points by

$$EP(f) = \limsup_{n \rightarrow +\infty} \frac{1}{n} \log \# P_n(f),$$

where $\#A$ is the cardinal number of a set A . In 1978, Bowen [11] asked the following question:

Question 1.1. Is the property that

$$EP = h$$

generic with respect to the C^r topology?

For Axiom A systems [8, 9, 10] one in fact has

$$EP = h.$$

Beyond uniform hyperbolicity, Katok [26] stated that, if f is a $C^{1+\alpha}$ diffeomorphism of a compact surface S with positive topological entropy, then $EP(f) \geq h(f)$. For any hyperbolic ergodic measure μ , the authors and Tian [22] established the equality between metric entropy and the exponential growth rate of those periodic measures approximating μ . Exactly in broad situations, due to the absence of uniform hyperbolicity, the periodic orbits can grow much faster than entropy. Linking with a conjecture of Palis [32], we mention two well known obstructions for the hyperbolicity: homoclinic tangencies [29] and heterodimensional cycles [1]. In [24] Kaloshin showed super-exponential growth of periodic orbits for a residual subset in some C^r -domain ($r \geq 2$) with persistent homoclinic tangencies. In [6] Bonatti, Díaz and Fisher proved super-exponential growth of periodic points for homoclinic classes with persistent heterodimensional cycles.

In the content of density, in 1965 Artin and Mazur [4] proved that: there exists a dense set \mathcal{D} of C^r maps such that for any map $f \in \mathcal{D}$, the number $\#P_n(f)$ grows at most exponentially with n (see also [25] for an extension concerning hyperbolic periodic points). For a vector field X , use ϕ_X to write the flow induced by X . Set

$$P_t(\phi_X) = \{ \text{isolated orb}(\phi_X, x) \mid \phi_X(x, 0) = \phi_X(x, s) \text{ for some } 0 \leq s \leq t \}$$

and define the exponential growth rate of periodic orbits by

$$EP(\phi_X) = \limsup_{t \rightarrow +\infty} \frac{1}{t} \log \# P_t(\phi_X).$$

Artin and Mazur [4] asked the following question for vector fields:

Question 1.2. Does the property that $EP(\phi_X) < \infty$ hold for a dense subset of $\mathcal{X}^r(M)$?

As we know, this question is far from being resolved, because the approaches concerning diffeomorphisms don't apply directly to flows. To continue the story of periodic orbits for dense systems we are in a position to understand more on the growth of periodic orbits of flows.

Two flows ϕ, ψ defined on a smooth manifold M are equivalent if there exists a homeomorphism π of M that sends each orbit of ϕ onto an orbit of ψ while preserving the time orientation :

$$\{\phi(x, t) \mid t \in \mathbb{R}\} = \{\pi^{-1}\psi(\pi(x), t) \mid t \in \mathbb{R}\}, \quad \forall x \in M.$$

Going back to the study of Lorenz attractors [20, 21] the kneading sequences were introduced to be invariants for equivalence. In general cases, it is not easy to find quantities preserved by equivalence. Topological entropy of a flow ϕ indicates, as usual, that for its time one map ϕ_1 , that is, $h(\phi) = h(\phi_1)$. Topological entropy is an invariant for equivalent homeomorphisms (Theorem 7.2 in [46]), while finite non-zero topological entropy for a flow cannot be an invariant because its value is affected by time reparameterization. For equivalent flows without fixed points the extreme value 0 and infinite entropy are invariant, while the sign of finite non-zero entropy are preserved (see [31], [35], [42], [43]). In equivalent flows with fixed points there exists a counterexample, constructed by Ohno [31], showing that neither 0 nor ∞ topological entropy is preserved by equivalence. The two flows constructed in [31] are suspensions of a transitive subshift and thus are not differentiable. Note that a differentiable flow on a compact manifold cannot have ∞ entropy (see Theorem 7.15 in [46]). Ohno [31] in 1980 asked the following:

Question 1.3. Is 0 topological entropy an invariant for equivalent differentiable flows?

In [39], Sun, Young and Zhou constructed two equivalent C^∞ flows with a singularity, one of which has positive topological entropy while the other has zero topological entropy. This gives a negative answer to Ohno's question.

Likewise as entropy, $EP = 0$ or $EP = \infty$ is invariant for equivalent homeomorphisms and also for equivalent flows without fixed points, see [31, 35]. For topological flows with fixed points, neither extreme growth rate, $EP = 0$ nor $EP = \infty$ is preserved for equivalence [40]. Moreover, there exists a pair of equivalent topological flows with fixed points such that one of which has ∞ topological entropy and 0 growth rate of periodic orbits but the other has 0 topological entropy and ∞ growth rate of periodic orbits [41].

In the present paper we are going to study in the world of smoothness and consider the following question:

Question 1.4. Is 0 or ∞ value of EP invariant for equivalent differentiable flows?

There are fruitful dynamical properties varying in the differentiability, for instance, symbolic extension which is exactly a suitable candidate to “measure” the dependence of entropy structure on the smoothness of underlying systems. Here we call a system (M, f) has a symbolic extension if there is a subshift (Y, g) over finite alphabets and a continuous surjection $\pi : Y \rightarrow M$ such that $f \circ \pi = \pi \circ g$.

$$\begin{array}{ccc} Y & \xrightarrow{g} & Y \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow{f} & M \end{array}$$

A symbolic extension (Y, g, π) for which $h_\nu(g) = h_\mu(f)$ for every g -invariant measure ν with $\pi_*\nu = \mu$ is viewed as a good model and is called a principal symbolic extension. In the context of C^∞ , Newhouse [30] showed upper semi-continuity of

metric entropy and Buzzi [14] further established asymptotical entropy expansiveness which, together with a criterion of Boyle, D. Fiebig and U. Fiebig [12] : if f is asymptotically entropy expansive, then f has a principal symbolic extension, implies all C^∞ maps admit a principal symbolic extension. In [17] Downarowicz and Newhouse constructed a G_δ set of C^1 area-preserving diffeomorphisms in which everyone has no symbolic extension. They described the entropy structure of C^2 differentiability by following:

Conjecture 1. Every C^2 map has a symbolic extension.

This conjecture has been proven by Downarowicz-Maass [16] for interval maps and by Burguet [13] for surface maps. Altogether, we have the following intuitions to reveal various differentiability:

C^1 differentiability \longleftrightarrow generically no symbolic extension

C^2 differentiability \longleftrightarrow symbolic extension (generically not principal)

C^∞ differentiability \longleftrightarrow principal symbolic extension.

We call a flow ϕ has a (principal) symbolic extension if its time one map $\phi(1, \cdot)$ has a (principal) symbolic extension. As we have stated, every C^∞ system has a principal symbolic extension. It is well known that symbolic systems with finite alphabets have finite EP . Our first theorem says that the finiteness of EP can't be inherited from its principal symbolic extension although the under system agrees the same entropy with the upper symbolic extension. This means that entropy is not enough to exhaust the difference of complexity even if in the category of C^∞ . Furthermore, we are going to show that the extreme growth rate of periodic orbits can't be preserved for orbit equivalent C^∞ flows.

Theorem A. *There exist two equivalent C^∞ flows φ and $\hat{\varphi}$ on the sphere \mathbb{S}^2 satisfying:*

$$EP(\varphi) = 0, \quad EP(\hat{\varphi}) = \infty.$$

Remark 1.5. Recall that L. S. Young [45] has proven zero entropy for all surface flows. However, much different from entropy Theorem A exhibits the existence of two dimensional flows with super-exponential growth of periodic orbits.

Remark 1.6. From our construction the statement of Theorem A can hold for any manifold of dimension ≥ 2 if a suitable embedding is taken. Here we only emphasis on the existence and thus omit details for general discussions.

Next we return to the relationship of EP and h for equivalent flows. As mentioned at beginning for almost (open dense) systems positive entropy enjoys the company of positive EP . In the next theorem we are close to be tightrope walkers since our examples are excluded by those open dense sets. We start by supposing that $f : M \rightarrow M$ is a C^∞ diffeomorphism of a smooth compact Riemannian manifold M with $\dim M = m \geq 4$ with the following properties: (1) f has positive topological entropy and (2) f is minimal in the sense that all forward orbits are dense in M (or equivalently closed invariant sets are either empty or the entire space). An example of such an f was constructed by Herman [23]. Using the constant function $I : M \rightarrow \mathbb{R}$, $I(x) = 1$, one gets a suspension manifold Ω and a

smooth vector field X associated with this flow. Since M is of dimension ≥ 4 , we know that $\dim \Omega \geq 5$.

Theorem B. *Given $K > 0$, there exist two C^2 equivalent flows ψ and $\hat{\psi}$ on Ω satisfying the following:*

- (1) $h(\psi) > K$ and $EP(\psi) = 0$;
- (2) $h(\hat{\psi}) = 0$ and $EP(\hat{\psi}) = \infty$.

2. TWO DIMENSIONAL EQUIVALENT FLOWS: PROOF OF THEOREM A

Throughout this section, we use D to denote the two-dimensional unit disk

$$\{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\}.$$

In order to obtain Theorem A, we proceed the main strategy of the proof as follows: firstly we arrange many enough periodic orbits with the same period on D and then we change their periods on different orbits according to the applications of different equivalent flows. Precisely, take a strictly decreasing sequence $\{a_i\}$ with $a_i = \frac{1}{i}$. For $i \geq 2$, let

$$l_i = \min\{a_i - a_{i+1}, a_{i-1} - a_i\},$$

$$b_{i,j} = a_i + \frac{j l_i}{2^{i+2}}, \text{ where } -2^{2^i} \leq j \leq 2^{2^i}.$$

Define the strips centered at the circle $r = a_i$ by

$$L_i = \{x \mid a_i - \frac{l_i}{4} \leq x \leq a_i + \frac{l_i}{4}\}.$$

Denote

$$I_0 = 0, \quad I_i = \sum_{j=1}^i 2^{2^j+1} + i, \quad i \geq 1.$$

Rearrange the sequence $\{b_{i,j}\} \cup \{1\}$ by decreasing order, denoting

$$b_1 = 1, \quad b_i = b_{s,j} \quad \text{for } i-1 = I_{s-1} + j, \quad |j| \leq 2^{2^s}.$$

For the purpose to get periodic orbits supporting on $r = b_i$, we give a C^∞ function α_0 on $[0, 1]$:

$$\alpha_0(x) = \begin{cases} e^{\frac{1}{(x-b_i^2)(x-b_{i+1}^2)}} & \text{for } b_i^2 < x < b_{i+1}^2, i \geq 1, \\ 0 & \text{for } x = 0 \text{ or } b_i^2, i \geq 1. \end{cases}$$

Consider a standard differential equation

$$\begin{cases} \frac{dx}{dt} = -y + \alpha_0(x^2 + y^2)x, \\ \frac{dy}{dt} = x + \alpha_0(x^2 + y^2)y. \end{cases}$$

Let $x = r \cos \theta, y = r \sin \theta$, then

$$\begin{cases} r \frac{dr}{dt} = \alpha_0(r^2)r^4, \\ \frac{d\theta}{dt} = 1. \end{cases}$$

For the sake of writing, denote the vector field $Z_0(x, y) = (-y + \alpha_0(x^2 + y^2)x, x + \alpha_0(x^2 + y^2)y)$. We can see that ϕ_{Z_0} has periodic orbits $r = b_i$ with the period 2π . To get different exponential growth rate of periodic orbits, next we will change the period for different i .

- (1) *Constriction of a flow on D with $EP = \infty$.*

Take a C^∞ function $\beta_1 : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\beta_1(x) = a_i^2 \quad \text{for } x \in L_i.$$

Consider the vector field $Z_1 = \beta_1(r^2)Z_0$, then the number of $2\pi n^2$ -periodic orbits is $2^{2^n+1} + 1$. Thus,

$$EP(\phi_{Z_1}) = \limsup_{t \rightarrow +\infty} \frac{1}{t} \log \# P_t(\phi_{Z_1}) \geq \limsup_{n \rightarrow +\infty} \frac{1}{2\pi n} \log (2^{2^n+1} + 1) = \infty.$$

(2) *Construction of a flow on D with $EP = 0$.*

Take a C^∞ function $\beta_2 : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\beta_2(x) = 2^{-2^i}, \quad \text{for } x \in L_i.$$

Consider the vector field $Z_2 = \beta_2(r^2)Z_0$, then the number of $2\pi 2^{2^i}$ -periodic orbits is $2^{2^i+1} + 1$. Thus,

$$\begin{aligned} EP(\phi_{Z_2}) &= \limsup_{t \rightarrow +\infty} \frac{1}{t} \log \# P_t(\phi_{Z_2}) \\ &= \limsup_{n \rightarrow +\infty} \frac{1}{2\pi 2^{2^n}} \log \left(\sum_{i=1}^n 2^{2^i+1} + 1 \right) \\ &\leq \limsup_{n \rightarrow +\infty} \frac{1}{2\pi 2^{2^n}} \log n(2^{2^n+1} + 1) \\ &= 0. \end{aligned}$$

Finally one can see that the two smooth flows ϕ_{Z_1} and ϕ_{Z_2} from our constructions are in fact equivalent.

Proof of Theorem A Let $\mathbb{S}^{2+} = \{(x_1, x_2, x_3) \in \mathbb{S}^2 \mid x_3 \geq 0\}$, $\mathbb{S}^{2-} = \{(x_1, x_2, x_3) \in \mathbb{S}^2 \mid x_3 \leq 0\}$. Define projection $\varrho(x_1, x_2, x_3) = (x_1, x_2)$ and $\varrho_+ = \varrho|_{\mathbb{S}^{2+}}$, $\varrho_- = \varrho|_{\mathbb{S}^{2-}}$. Next we embed the flows ϕ_{Z_i} of D into \mathbb{S}^2 by the double cover ϱ .

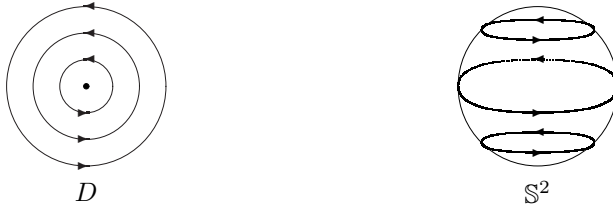


FIGURE 1. Flows on D and \mathbb{S}^2

Precisely, let

$$\begin{aligned} \varphi &= \begin{cases} \varrho_-^{-1} \circ \phi_{Z_2} \circ \varrho_- & \text{for } x \in \mathbb{S}^{2-}, \\ \varrho_+^{-1} \circ \phi_{Z_2} \circ \varrho_+ & \text{for } x \in \mathbb{S}^{2+}. \end{cases} \\ \widehat{\varphi} &= \begin{cases} \varrho_-^{-1} \circ \phi_{Z_1} \circ \varrho_- & \text{for } x \in \mathbb{S}^{2-}, \\ \varrho_+^{-1} \circ \phi_{Z_1} \circ \varrho_+ & \text{for } x \in \mathbb{S}^{2+}. \end{cases} \end{aligned}$$

Then φ and $\widehat{\varphi}$ are equivalent flows on \mathbb{S}^2 with the desired property:

$$EP(\varphi) = 0, \quad EP(\widehat{\varphi}) = \infty.$$

□

3. HIGH DIMENSIONAL EQUIVALENT FLOWS: PROOF OF THEOREM B

3.1. Basic notions and technique lemmas on suspension flows.

Before the construction, we need do some preliminaries.

Let $(M, \mathcal{B}(M), \mu)$ be a probability space and f be a μ -measurable map.

Consider the space $\Omega = M \times [0, 1] / \sim$, where \sim is the identification of $(y, 1)$ with $(f(y), 0)$. The standard suspension of f is the flow ϕ on Ω defined by $\phi(y, s) = (y, t + s)$, for $0 \leq t + s < 1$. A standard argument as in [27] shows that Ω is a C^∞ smooth compact Riemannian manifold and ϕ is C^∞ provided $f : M \rightarrow M$ is a C^∞ diffeomorphism on C^∞ smooth manifold M . If $f : M \rightarrow M$ is minimal as a homeomorphism, then ϕ is a minimal flow.

Proposition 3.1 (Proposition 2.15 of [39]). *Let μ be an invariant ergodic measure of f on M . We define*

$$\int_{\Omega} g d\overline{\mu} := \int_E \int_0^1 g(x, t) dt d\mu, \quad \forall g \in C^0(\Omega).$$

Then we have

$$h_{\overline{\mu}}(\psi) = h_{\mu}(f).$$

Definition 3.2. Suppose ϕ is a measurable flow on a Borel probability space $(M, \mathcal{B}(M), \nu)$ and Ω is divided into disjoint invariant measurable sets F_1 and F_2 such that $\mu(F_1) = 1$ and $\mu(F_2) = 0$. Further suppose that $\theta(t, x)$ is a real measurable function defined on $(-\infty, +\infty) \times (\Omega \setminus F_2) = \mathbb{R} \times F_1$ with the following properties for every fixed $x \in F_1$:

- (1) $\theta(t, x)$ is continuous and non-decreasing in t ;
- (2) $\theta(t + s, x) = \theta(s, x) + \theta(t, \phi_s(x))$ for all t and s ;
- (3) $\theta(0, x) = 0$, $\lim_{t \rightarrow +\infty} \theta(t, x) = \infty$, $\lim_{t \rightarrow -\infty} \theta(t, x) = -\infty$.

Then θ is called an additive function of Ω with carrier F_2 . An additive function is said to be integrable if it is integrable in Ω for every fixed t .

For a non-negative, integrable function $a(x)$, we define

$$E_{\mu}(a) = \int_E a(x) d\mu(x).$$

Lemma 3.3. *If ϕ is a measurable flow on a Borel probability space $(\Omega, \mathcal{B}(\Omega), \nu)$ and $a(x)$ is a non-negative, integrable function satisfying*

$$E_{\mu}(a) = \int_{\tilde{E}} a(x) d\mu(x) > 0,$$

then the function

$$\theta(t, x) = \int_0^t a(\phi_s(x)) ds$$

is an integrable additive function.

For a proof see Theorem 3.1 in [44].

Definition 3.4. The function $\theta(t, x)$ in Lemma 3.3 is called the additive function defined by $a(x)$.

Lemma 3.5. *Let μ be an invariant probability measure of f on E . Assume $\theta(t, x)$ is the additive function defined by $a(x)$ with $0 < E_\mu(a) < \infty$. We define*

$$\int_{\Omega} g d\hat{\mu} := \frac{1}{E_\mu(a)} \int_E \int_0^{\theta(x,t)} g(x, t) dt d\mu, \quad \forall g \in C^0(\Omega).$$

Then $\hat{\mu}$ is an invariant measure of ϕ_t on Ω . Further, $\hat{\mu}$ is ergodic if μ is ergodic.

The proof is elementary and omitted.

Lemma 3.6 (Lemma 2.4 of [39]). *Suppose (M, f) is a minimal homeomorphism. Then for any $\varepsilon > 0$, there exists $L(\varepsilon) > 0$ such that for any ergodic invariant measure μ , we have*

$$\mu(B_M(x, \varepsilon)) \geq \frac{1}{L(\varepsilon)} > 0, \quad \forall x \in M.$$

Lemma 3.7 (Corollary 2.12 of [39]). *Assume that ϕ_t is the standard suspension of a minimal homeomorphism (M, f) from above, X is the vector field that induces ϕ_t and $\alpha \in C^1(M, [0, 1])$. Denote by $\phi_{\alpha X}$ the flow induced by the vector field αX on Ω . For any $x \in M$, define $\gamma(x)$ by:*

$$\begin{cases} \phi_{\alpha X}((x, 0), \gamma(x)) = \phi_{\alpha X}((x, 0), 1) = (f(x), 0), & (x, 0) \neq f^{-1}(p) \text{ and } (x, 0) \neq p; \\ \gamma(x) = +\infty, & (x, 0) = f^{-1}(p) \text{ or } (x, 0) = p. \end{cases}$$

If $E_\mu(\gamma) = +\infty$ for any non-atomic ergodic measure μ of f , then $\phi_{\alpha X}$ has only atomic invariant Borel probability measures.

3.2. Proof of Theorem B.

Step 1 Construction of a flow with zero entropy.

Take $p_0 = [(x_0, 0)] = \pi(x_0, 0)$ where π is the quotient map $\pi : M \times \mathbb{R} \rightarrow \Omega$.

Without loss of generality, we can assume the existence of a coordinate chart (\tilde{V}, ξ) of Ω satisfying the following:

- (i) There exists an open set V of Ω , such that $p_0 \in V$ and $\overline{V} \subset \tilde{V}$.
- (ii) $\xi(p_0) = 0, \xi(V) = B^{m+1}(0, 1), \xi(\tilde{V}) = B^{m+1}(0, 2)$, where $2 \leq m = \dim M$.
- (iii) There exists $i_1 \in \mathbb{N}$ such that

$$\text{cl}(\pi(B_M(x_0, i_1^{-1}) \times \{0\})) \subset V$$

and

$$\xi(\pi(B_M(x_0, i_1^{-1}) \times \{0\})) \subset \mathcal{R} = \{x = (x_1, \dots, x_m, x_{m+1}) : x_{m+1} = 0\},$$

where $\text{cl}(F)$ denotes the closure of a subset $F \subset \Omega$.

- (iv) $\exists i_2 \in \mathbb{N}$ such that

$$B_\Omega(p_0, i_2^{-1}) \subset V$$

and

$$\xi(B_\Omega(p_0, i_2^{-1})) = B^{m+1}(0, i_2^{-1})$$

for any $i_2 < i \in \mathbb{N}$. Under these assumptions, there exists $i_3 \in \mathbb{N}$ and $i_2 < i_3$, with the property that for any $i \geq 0$ there exists $1 \gg l_{i_3+i} > 0$ such that

$$\text{cl}(\pi(B_M(x_0, \frac{1}{i_3+i}) \times [-l_{i_3+i}, 0])) \subset B_\Omega(p, \frac{1}{i_2+i}).$$

We set $i_0 := \max\{i_1, i_2, i_3\}$. For any $i > i_0$, by Lemma 3.6, there exists $L(\frac{1}{i})$ such that for any ergodic measure τ of f , we have

$$\tau(B_M(f^{-1}x_0, \frac{1}{i})) \geq \frac{1}{L(\frac{1}{i})} := \delta(i) > 0.$$

We define $\beta_{-1} := 1$ and $\beta_{i-1} := \frac{l_{i_0+i}}{i_0+i} \delta(i_0+i)$ for $i \geq 1$. We need the following Lemma to construct satisfactory smooth vector fields.

Lemma 3.8. *For a given sequence of positive numbers $1 = \beta_0 > \beta_1 > \beta_2 > \dots > \beta_i > \dots$, there exists a C^∞ function $w : B^{m+1}(0, 2) \rightarrow [0, 1]$ such that*

- (1) $w = 0$ if and only if $x = 0$;
- (2) $\|w|_{B^{m+1}(\frac{1}{i+1})}\|_{C^0} \leq \beta_{i-1}$, $i = 0, 1, 2, \dots$;
- (3) $w|_{B^{m+1}(2) \setminus B^{m+1}(1)} = 1$.

Proof. Without loss of generality, we assume that $\lim_{i \rightarrow \infty} \beta_i = 0$. Let $\Psi(t)$ be the function:

$$\Psi(t) = \begin{cases} e^{-\frac{1}{t}}, & 0 < t \leq 1, \\ 0, & -1 < t \leq 0. \end{cases}$$

Let $\{\beta_i\}$ be as above and suppose c_i is any decreasing sequence of positive numbers $1 = c_{-1} > c_0 > c_1 > \dots > c_i > \dots$, with $\lim_{i \rightarrow \infty} c_i = 0$. For $t < c_0$, let $g(t)$ be the function on $[-1, c_0]$ defined by the series:

$$g(t) = \sum_{i=1}^{\infty} 2^{-i-1} \beta_{i-1} \Psi(t - c_i).$$

This series is monotone increasing in i and converges uniformly. It is zero on $[-1, 0]$ and positive on $(0, c_0]$. For any k and $0 < t < c_k$ we have

$$\sum_{i=k+1}^{\infty} 2^{-i-1} \beta_{i-1} \Psi(t - c_i) < \frac{\beta_k \Psi(1)}{2^k}.$$

Further, since the derivatives of the partial sums of this series converge uniformly, we may take the derivative of the sum and we have that:

$$g'(t) = \sum_{i=1}^{\infty} 2^{-i-1} \beta_{i-1} \Psi'(t - c_i) < \infty.$$

By induction, we may also conclude that

$$g^{(l)}(t) = \sum_{i=1}^{\infty} 2^{-i-1} \beta_{i-1} \Psi^{(l)}(t - c_i)$$

converges uniformly and

$$\lim_{t \rightarrow 0} g^{(l)}(t) = 0.$$

We can now clearly extend $g(t)$ to the interval $[0, 2]$ in such a way that g is C^∞ and $g(t) = 1$ for $t \in [1, 2]$. To finish the proof of the lemma, we set $c_i = \frac{1}{i+1}$ in

the above construction of $g(t)$ and let $w(x) = g(|x|)$ on $B^{m+2}(2)$. Because of the construction this function is C^∞ smooth at 0 and on $B^{m+2}(2)$. \square

Using Lemma 3.8, one can find a C^∞ function $\omega_1 : \xi(\tilde{V}) \rightarrow [0, 1]$ with the properties:

- (i) $\omega_1|_{\xi(\tilde{V} \setminus V)} \equiv 1$;
- (ii) $\|\omega_1|_{B^{m+1}(0, \frac{1}{i_2+i})}\| \leq \beta_{i-1}$;
- (iii) $\omega_1(0) = 0$ and $0 < \omega_1(a) \leq 1$ for $0 \neq a \in \xi(\tilde{V})$.

We then define a function $\alpha \in C^\infty(\Omega, [0, 1])$ as follows:

$$\alpha(q) := \begin{cases} \omega_1 \circ \xi(q), & q \in \tilde{V}; \\ 1, & q \in \Omega \setminus \tilde{V}. \end{cases}$$

Then

$$\|\alpha|_{B_\Omega(p, \frac{1}{i_2+i})}\|_{C^0} = \sup_{x \in B_\Omega(p, \frac{1}{i_2+i})} \{\alpha(x)\} \leq \beta_{i-1},$$

where we assume, without loss of generality, that $\|\xi\|_{C^0} \leq 1$. We then define $Y := \alpha X$ and let ϕ_t denote the flow induced by Y . Recall the function $\gamma : M \rightarrow R \cup \{\infty\}$ in Lemma 3.8 and observe that for any $x \in B_M(f^{-1}(x_0), \frac{1}{i_2+i})$,

$$l_{i_0+i} = \int_{t(x)}^{\gamma(x)} \sqrt{\langle \alpha(\phi_s(x)X(\phi_s(x))), \alpha(\phi_s(x)X(\phi_s(x))) \rangle} ds,$$

where $t(x) > 0$ satisfies $\phi_{t(x)}(x) = \phi_{1-l_{i_0+i}}(x)$. Then

$$\gamma(x) \geq \gamma(x) - t(x) \geq \frac{l_{i_0+i}}{\|\alpha|_{B_\Omega(p, \frac{1}{i_2+i})}\| \|X\|} \geq \frac{l_{i_0+i}}{\beta_{i-1} \|X\|} = \frac{i_0+i}{\delta(i_0+i) \|X\|}$$

for any $x \in B_M(f^{-1}(x_0), \frac{1}{i_0+i})$. Thus,

$$\gamma|_{B_M(f^{-1}(x_0), \frac{1}{i_0+i})} \geq \frac{i_0+i}{\delta(i_0+i) \|X\|}.$$

For any ergodic measure ν of f ,

$$E(\gamma) = \int_M \gamma(x) d\nu(x) \geq \frac{i_0+i}{\delta(i_0+i) \|X\|} \nu(B_M(f^{-1}(x_0), \frac{1}{i_0+i})) \geq \frac{i_0+i}{\|X\|} \rightarrow +\infty,$$

as $i \rightarrow +\infty$. So, by Lemma 3.7 all ergodic measures of $\phi_{\alpha X}$ are atomic, which implies $h(\phi_{\alpha X}) = 0$. In fact, $\phi_{\alpha X} = \phi_t$ has only one invariant measure δ_{p_0} .

Take another point $p_1 \in \Omega \setminus \text{cl}(\tilde{V})$. Choose a suitable smooth coordinate chart (\tilde{U}, ζ) of Ω satisfying that

- (i) there is an open set $U \subset \text{cl}(U) \subset \tilde{U} \subset \text{cl}(\tilde{U}) \subset \Omega \setminus \text{cl}(\tilde{V})$.
- (ii) $\zeta(p_1) = 0$, $\zeta(U) = B^{m+1}(0, 4)$, $\zeta(\tilde{U}) = B^{m+1}(0, 8)$.
- (iii) the flow $\psi_0 := \zeta \circ \phi_X \circ \zeta^{-1}|_{\zeta(\tilde{U})}$ induced by the following standard differential equation

$$\begin{cases} \dot{x}_1 = 1; \\ \dot{x}_2 = 0; \\ \vdots \\ \dot{x}_{m+1} = 0. \end{cases}$$

We select a C^2 function $\eta : \mathbb{R}^{m+1} \rightarrow \mathbb{R}$ satisfying

(1)

$$\eta(x) = \begin{cases} (\sum_{i=2}^{m+1} x_i^2)^{\frac{3}{2}} & x \in B^{m+1}(0, 2), -1 \leq x_1 \leq 1, \\ (\sum_{i=2}^{m+1} x_i^2)^{\frac{3}{2}} + e^{\frac{1}{x_1^2-1}} & x \in B^{m+1}(0, 2), x_1 > 1 \text{ or } x_1 < -1; \end{cases}$$

(2) $\frac{1}{20} < \eta(x) < 20$, when $2 \leq \|x\| \leq 4$.(3) $\eta(x) = 1$, when $\|x\| \geq 4$.

Noticing that $m+1 \geq 5$, it is easy to verify that $\frac{1}{\eta(x)}$ is integrable with respect to Lebesgue measure on $B^{m+1}(0, 8)$. Define a function $\alpha_1 : \Omega \rightarrow \mathbb{R}$ as follows:

$$\alpha_1(q) = \begin{cases} \omega_1 \circ \xi^{-1}(q) & q \in \tilde{V}; \\ \eta \circ \zeta^{-1}(q) & q \in \tilde{U}; \\ 1 & q \in \Omega \setminus (\tilde{U} \cup \tilde{V}). \end{cases}$$

Then $\alpha_1 X$ induces a flow $\phi_{\alpha_1 X}$ on Ω . Denote

$$F_0 = \{x \in B^{m+1}(0, 4) \mid -1 \leq x_1 \leq 1, \quad x_2 = \cdots = x_{m+1} = 0\}.$$

Then all points contained in F_0 are singularities of $\phi_{\alpha_1 X}$. In what follows, we will show that the integration of $\frac{1}{\eta(x)}$ can guarantee the modifications in \tilde{U} don't contribute entropy of the consequent flows.

Proposition 3.9. $h(\phi_{\alpha_1 X}) = 0$.

Proof. Given a flow ϕ on Ω , for any Borel set $B \in \mathcal{B}(\Omega)$, $p \in \Omega$ and $t > 0$, define

$$I(t, p, \phi, B) = \{0 \leq s \leq t \mid \phi(p, s) \in B\}.$$

and

$$J(t, p, \phi, B) = \text{Leb}(I(t, p, \phi, B)) = \int_0^t \chi_B(\phi(p, s)) ds,$$

where

$$\chi_B(x) = \begin{cases} 1 & x \in B, \\ 0 & x \in \Omega \setminus B. \end{cases}$$

Arbitrarily taking an open set $U_0 \in \Omega \setminus (\tilde{U} \cup \tilde{V})$, since $\phi_{\alpha X}$ has only one invariant probability measure δ_{p_0} , for any $p \in \Omega \setminus (\tilde{U} \cup \tilde{V})$ we have

$$\lim_{t \rightarrow +\infty} \frac{J(t, p, \phi_{\alpha X}, U_0)}{t} = 0.$$

By definition of η one can obtain $\|\eta\| \leq 20$, which implies that

$$\begin{aligned} J(t, p, \phi_{\alpha_1 X}, \tilde{U}) &= \int_{I(t, p, \phi_{\alpha X}, \tilde{U})} \frac{1}{\alpha_1(\phi_{\alpha X}(p, s))} ds \\ &\geq \int_{I(t, p, \phi_{\alpha X}, \tilde{U})} \frac{1}{20} ds \\ &= \frac{1}{20} J(t, p, \phi_{\alpha X}, \tilde{U}). \end{aligned}$$

Define $\lambda(t) = J(t, p, \phi_{\alpha_1 X}, \tilde{U}) + J(t, p, \phi_{\alpha X}, \Omega \setminus \tilde{U})$. Noting that $\alpha_1(q) = \alpha(q)$ for $q \in \Omega \setminus \tilde{U}$, we have

$$J(\lambda(t), p, \phi_{\alpha_1 X}, U_0) = J(t, p, \phi_{\alpha X}, U_0).$$

Therefore,

$$\begin{aligned} \frac{1}{\lambda(t)} \int_0^{\lambda(t)} \chi_{U_0}(\phi_{\alpha_1 X}(p, s)) ds &= \frac{J(t, p, \phi_{\alpha X}, U_0)}{J(t, p, \phi_{\alpha X}, \tilde{U}) + J(t, p, \phi_{\alpha X}, \Omega \setminus \tilde{U})} \\ &\leq \frac{J(t, p, \phi_{\alpha X}, U_0)}{\frac{1}{20}J(t, p, \phi_{\alpha X}, \tilde{U}) + J(t, p, \phi_{\alpha X}, \Omega \setminus \tilde{U})}. \end{aligned}$$

Moreover,

$$\begin{aligned} \lim_{t \rightarrow +\infty} \frac{J(t, p, \phi_{\alpha X}, U_0)}{t} &= \lim_{t \rightarrow +\infty} \frac{J(t, p, \phi_{\alpha X}, \tilde{U})}{t} = 0, \\ \lim_{t \rightarrow +\infty} \frac{J(t, p, \phi_{\alpha X}, \Omega \setminus \tilde{U})}{t} &= 1. \end{aligned}$$

We deduce that

$$\begin{aligned} &\limsup_{t \rightarrow +\infty} \frac{1}{\lambda(t)} \int_0^{\lambda(t)} \chi_{U_0}(\phi_{\alpha_1 X}(p, s)) ds \\ &\leq \limsup_{t \rightarrow +\infty} \frac{J(t, p, \phi_{\alpha X}, U_0)}{\frac{1}{20}J(t, p, \phi_{\alpha X}, \tilde{U}) + J(t, p, \phi_{\alpha X}, \Omega \setminus \tilde{U})} \\ &= \limsup_{t \rightarrow +\infty} \frac{\frac{1}{t}J(t, p, \phi_{\alpha X}, U_0)}{\frac{1}{20t}J(t, p, \phi_{\alpha X}, \tilde{U}) + \frac{1}{t}J(t, p, \phi_{\alpha X}, \Omega \setminus \tilde{U})} \\ &= 0. \end{aligned}$$

So, for any ergodic invariant measure ν of $\phi_{\alpha_1 X}$, $\text{supp}(\nu) \cap U_0 = \emptyset$. All ergodic measures are supported on $\tilde{U} \cup \tilde{V}$. Besides, all ergodic measures in \tilde{U} are atomic. Thus $h(\phi_{\alpha_1 X}) = 0$. \square

Step 2 Construction of a flow with positive entropy.

We select a smooth function $\hat{\omega}_1 : \mathbb{R}^{m+1} \rightarrow \mathbb{R}$ satisfying

- (1) $\hat{\omega}_1(x) = \|x\|^2 = \sum_{i=1}^{m+1} x_i^2$, when $\|x\| \leq \frac{1}{2}$;
- (2) $\frac{1}{4} < \hat{\omega}_1(x) < 2$, when $\frac{1}{2} < \|x\| < 1$;
- (3) $\hat{\omega}_1(x) = 1$, when $\|x\| \geq 1$.

We can see that $\frac{1}{\hat{\omega}_1(x)}$ is integrable with respect to Lebesgue measure on $B^{m+1}(0, 2)$ since $m+1 \geq 3$.

Define a new smooth function $\hat{\alpha}_1 : \Omega \rightarrow \mathbb{R}$ as follows:

$$\hat{\alpha}_1(p) = \begin{cases} \hat{\omega}_1 \circ \xi^{-1}(p) & p \in \tilde{V}; \\ \eta \circ \zeta^{-1}(p) & p \in \tilde{U}; \\ 1 & p \in \Omega \setminus (\tilde{U} \cup \tilde{V}). \end{cases}$$

Then

$$a_1 := \int_{\Omega} \frac{1}{\hat{\alpha}_1(p)} d\bar{\mu}(p) < \infty.$$

As appoint before, denote by $\phi_{\hat{\alpha}_1 X}$ the flow induced by $\hat{\alpha}_1 X$. Recalling Herman's example, f preserves an ergodic measure μ equivalent to the Riemannian volume. In Herman's construction, for any $N \in \mathbb{N}$, f^N also preserves the volume measure μ and is minimal. So for any $K_1 > 0$ we can take N large so that $h_{\mu}(f^N) = Nh_{\mu}(f) >$

$a_1 K_1$ (Here we don't change the construction of the manifold Ω). Without loss of generality, in what follows we assume $h_\mu(f) > a_1 K_1$.

Proposition 3.10. $h(\phi_{\widehat{\alpha}_1 X}) > K_1$.

Proof. First Lemma 3.5 and Lemma 3.1 apply to give that $h_{\bar{\mu}}(\phi_X) = h_\mu(f) > a_1 K_1$. Define a new measure $\widehat{\mu}_1$ on Ω as follows

$$\widehat{\mu}_1(B) = \int_B d\widehat{\mu}_1(x) = \int_B \frac{1}{\widehat{\alpha}_1(x)} d\bar{\mu}(x)$$

for all $B \in \mathcal{B}(\Omega)$. By Lemma 3.5, $\widehat{\mu}_1$ is an invariant ergodic measure of $\phi_{\widehat{\alpha}_1 X}$ and

$$\widehat{\mu}_1(\Omega) \leq \int_\Omega \frac{1}{\widehat{\alpha}_1(x)} d\bar{\mu}(x) < \infty.$$

Noting that μ is ergodic, by Theorem 3.1 we have

$$h_{\widehat{\mu}_1}(\phi_{\widehat{\alpha}_1 X}) \widehat{\mu}_1(\Omega) = h_{\bar{\mu}}(\phi_X) \bar{\mu}(\Omega)$$

which gives rise to $h_{\widehat{\mu}_1}(\phi_{\widehat{\alpha}_1 X}) \geq \frac{h_{\bar{\mu}}(\phi_X)}{a_1} > K_1$. By the variational principle (see for example §8.2 of [46]) we conclude that

$$h(\phi_{\widehat{\alpha}_1 X}) \geq h_{\widehat{\mu}_1}(\phi_{\widehat{\alpha}_1 X}) > K_1.$$

□

Proposition 3.11. $\phi_{\alpha_1 X}$ and $\phi_{\widehat{\alpha}_1 X}$ are equivalent.

Proof. The identity map on Ω takes orbits of one flow $\phi_{\alpha_1 X}$ to orbits of the other $\phi_{\widehat{\alpha}_1 X}$ since the singular points p_0, p_1 mapped to themselves and elsewhere α_1 and $\widehat{\alpha}_1$ are positive. The assumption that α_1 and $\widehat{\alpha}_1$ are non-negative also implies preservation of time orientation and hence the equivalence of $\phi_{\alpha_1 X}$ and $\phi_{\widehat{\alpha}_1 X}$. □

Step 3 Tear the segment A_0 to be ball-like.

We begin by choosing a smooth function $\gamma_0 : \mathbb{R} \rightarrow \mathbb{R}$ as follows

$$\gamma_0(x) = \begin{cases} 0 & x \leq -1; \\ e^{\frac{1}{x^2-1}} & -1 < x < 1; \\ 0 & x \geq 1. \end{cases}$$

Clearly $\gamma_0(x)$ is C^∞ smooth and has zero derivatives of all orders at -1 and 1 . Then consider two families of C^∞ curves:

$$\begin{aligned} \rho_a(s) &= (s, a\gamma_0(s)) \quad 0 \leq a \leq 1; \\ \sigma_b(s) &= \begin{cases} (s, \gamma_0(s) + b) & 0 \leq b \leq 1; \\ (s, (2-b)(\gamma_0(s) + 1) + 2(b-1)) & 1 \leq b \leq 2, \end{cases} \end{aligned}$$

where $-2 < s < 2$. Direct computations give rise to the following expressions:

$$\begin{aligned} \frac{d\rho_a(x_1)}{dx_1} &= (1, a\gamma'_0(x_1)) \quad 0 \leq a < 1, \\ \frac{d\sigma_b(x_1)}{dx_1} &= \begin{cases} (1, \gamma'_0(x_1)) & 0 \leq b < 1; \\ (1, (2-b)\gamma'_0(x_1)) & 1 \leq b \leq 2. \end{cases} \end{aligned}$$

Denote regions

$$U_1 = \{(x_1, x_2) \mid x_2 = \rho_a(x_1), -2 < x_1 < 2, 0 \leq a \leq 1\},$$

$$V_1 = \{(x_1, x_2) \mid x_2 = \sigma_b(x_1), -2 < x_1 < 2, 0 \leq b \leq 1\},$$

$$W_1 = \{(x_1, x_2) \mid x_2 = \sigma_b(x_1), -2 < x_1 < 2, 1 \leq b \leq 2\}.$$

Now we define a projection $\Delta : U_1 \cup V_1 \cup W_1 \rightarrow U_1 \cup V_1 \cup W_1$ as follows

$$\begin{aligned} \Delta(x) &= (x_1, 0) & \text{if } x = (x_1, x_2) \in \rho_a; \\ \Delta(x) &= (x_1, b) & \text{if } x = (x_1, x_2) \in \sigma_b. \end{aligned}$$

Obviously Δ is continuous. Moreover,

$$\Delta|_{\rho_a} : \rho_a \rightarrow (-2, 2) \times \{0\}, \quad \Delta|_{\sigma_b} : \sigma_b \rightarrow (-2, 2) \times \{b\}$$

are both onto C^∞ diffeomorphisms. For every $1 \leq k \leq m$, consider the standard embedding $\varrho_k : \mathbb{R}^k \rightarrow \mathbb{R}^{m+1}$ given by

$$\varrho_k(x_1, x_2, \dots, x_k) = (x_1, x_2, \dots, x_k, 0, \dots, 0).$$

The projection π_k is defined by

$$\pi_k(x_1, x_2, \dots, x_k, x_{k+1}, \dots, x_{m+1}) = x_k.$$

Let $\varphi_0(y, t) := \zeta \circ \varphi_{\alpha_1 X}(\zeta^{-1}(y), t)$ for $y \in B^{m+1}(0, 8)$. Denote a restricted flow $\widehat{\phi}$ on $U_1 \cup V_1 \cup W_1$

$$\widehat{\phi}((x_1, x_2), t) := (\pi_1 \circ \varphi_0(\varrho_2((x_1, x_2))), t, \pi_2 \circ \varphi_0(\varrho_2((x_1, x_2))), t).$$

Now we define a flow φ_1 on $U_1 \cup V_1 \cup W_1$ as follows

$$\varphi_1|_{\rho_a} = (\Delta|_{\rho_a})^{-1} \circ \widehat{\phi} \circ (\Delta|_{\rho_a}).$$

$$\varphi_1|_{\sigma_b} = (\Delta|_{\sigma_b})^{-1} \circ \widehat{\phi} \circ (\Delta|_{\sigma_b}).$$

Next, we calculate the expression of the vector field $Z(x)$ associated with the flow φ_1 .

Case 1: $x = (x_1, x_2) \in \rho_a$. $a\gamma_0(x_1) = x_2$, so $a = \frac{x_2}{\gamma_0(x_1)}$. It follows that

$$\begin{aligned} \frac{d\varphi_1(x, t)}{dt} \Big|_{t=0} &= d((\Delta|_{\rho_a})^{-1}) \frac{d}{dt} \widehat{\phi} \circ (\Delta|_{\rho_a}) \Big|_{t=0} \\ &= d((\Delta|_{\rho_a})^{-1}) \frac{d}{dt} \widehat{\phi}((x_1, 0), t) \Big|_{t=0} \\ &= \left(\frac{d}{dt} \widehat{\phi}((x_1, 0), t) \Big|_{t=0}, u(a)\gamma_0'(x_1) \frac{d}{dt} \widehat{\phi}((x_1, 0), t) \Big|_{t=0} \right) \\ &= \eta(\varrho_1(x_1))(1, a\gamma_0'(x_1)); \end{aligned}$$

Case 2: $x = (x_1, x_2) \in \sigma_b$ for $0 \leq b < 1$. $x_2 = \gamma_0(x_1) + b$, so $b = x_2 - \gamma_0(x_1)$.

$$\begin{aligned} \frac{d\varphi_1(x, t)}{dt} \Big|_{t=0} &= d((\Delta|_{\sigma_b})^{-1}) \frac{d}{dt} \widehat{\phi} \circ (\Delta|_{\sigma_b}) \Big|_{t=0} \\ &= d((\Delta|_{\sigma_b})^{-1}) \frac{d}{dt} \widehat{\phi}((x_1, b), t) \Big|_{t=0} \\ &= \eta(\varrho_2(x_1, x_2 - \gamma_0(x_1)))(1, \gamma_0'(x_1)); \end{aligned}$$

Case 3: $x = (x_1, x_2) \in \sigma_b$ for $1 \leq b \leq 2$. $x_2 = (2 - b)(\gamma_0(x_1) + 1) + 2(b - 1)$.

$$\begin{aligned} \frac{d\varphi_1(x, t)}{dt} \Big|_{t=0} &= \eta(\varrho_2(x_1, b))(1, (2 - b)\gamma'_0(x_1)) \\ &= \eta(\varrho_2(x_1, \frac{x_2 - 2\gamma_0(x_1)}{1 - \gamma_0(x_1)}))(1, \frac{2 - x_2}{1 - \gamma_0(x_1)}\gamma'_0(x_1)). \end{aligned}$$

Hence,

$$Z(x) = \begin{cases} \eta(\varrho_1(x_1))(1, a\gamma'_0(x_1)), & x \in U_1; \\ \eta(\varrho_2(x_1, x_2 - \gamma_0(x_1)))(1, \gamma'_0(x_1)), & x \in V_1; \\ \eta(\varrho_2(x_1, \frac{x_2 - 2\gamma_0(x_1)}{1 - \gamma_0(x_1)}))(1, \frac{2 - x_2}{1 - \gamma_0(x_1)}\gamma'_0(x_1)), & x \in W_1; \\ \eta(\varrho_2(x_1, x_2))(1, 0) & x \in \mathbb{R}^2 \setminus (U_1 \cup V_1 \cup W_1). \end{cases}$$

It is verified that $Z(x)$ is C^2 with respect to $x \in U_1 \cup V_1$. Furthermore, $Z(x) = 0$ for all $x = (x_1, x_2) \in U_1$ with $-1 \leq x_1 \leq 1$.

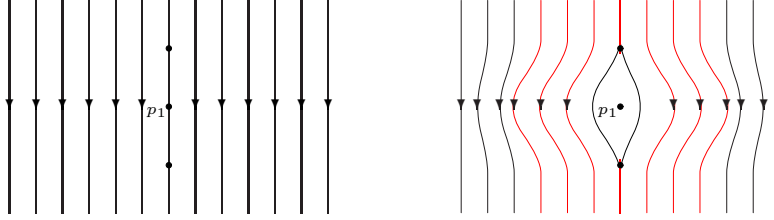


FIGURE 2. Flows $\hat{\phi}$ and φ_1 on $U_1 \cup V_1 \cup W_1$

In order to get a global flow on Ω , we use the usual $(m+1)$ -dimensional orthogonal group, denoted by $O(m)$, to rotate Z . For any $A \in O(m)$, define

$$\tilde{A} = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}.$$

Let $\tilde{U}_1 = O(m)\varrho_2 U_1$, $\tilde{V}_1 = O(m)\varrho_2 V_1$, $\tilde{W}_1 = O(m)\varrho_2 W_1$. Given $x = (x_1, \dots, x_{m+1}) = \tilde{A} \circ \varrho_2(x_1, \sqrt{\sum_{i=2}^{m+1} x_i^2}) \in B^{m+1}(0, 4)$ for some $A \in O(m)$, denote the vector field \tilde{Z} as the rotation of Z with the precise form

$$\tilde{Z}(x) = \frac{d}{dt} \Big|_{t=0} \tilde{A} \circ \varrho_2 \circ \varphi_1((x_1, \sqrt{\sum_{i=2}^{m+1} x_i^2}), t) = d(\tilde{A} \circ \varrho_2)Z((x_1, \sqrt{\sum_{i=2}^{m+1} x_i^2})).$$

Together with the construction of Z , it follows that

$$\tilde{Z}(x) = \begin{cases} \eta(\varrho_1(x_1))(1, 0, \dots, 0), & (x_1, \sqrt{\sum_{i=2}^{m+1} x_i^2}) \in U_1; \\ \eta(\varrho_2(x_1, \sqrt{\sum_{i=2}^{m+1} x_i^2} - \gamma(x_1)))(1, \gamma'(x_1) \frac{x_2}{\sqrt{\sum_{i=2}^{m+1} x_i^2}}, \dots, \gamma'(x_1) \frac{x_{m+1}}{\sqrt{\sum_{i=2}^{m+1} x_i^2}}), & (x_1, \sqrt{\sum_{i=2}^{m+1} x_i^2}) \in V_1. \end{cases}$$

Then $\tilde{Z}(x)$ is C^2 on $\tilde{U}_1 \cup \tilde{V}_1$ and the corresponding flow $\phi_{\tilde{Z}}$ is given by

$$\phi_{\tilde{Z}}(x, t) = \tilde{A} \circ \varrho_2 \circ \varphi_1((x_1, \sqrt{\sum_{i=2}^{m+1} x_i^2}, t).$$

Now we define $\tilde{\pi} : \tilde{U}_1 \cup \tilde{V}_1 \cup \tilde{W}_1 \rightarrow \tilde{U}_1 \cup \tilde{V}_1 \cup \tilde{W}_1$ as follows

$$\begin{aligned} \tilde{\pi}(x) &= \varrho_1(x_1), \text{ if } x \in \tilde{A} \circ \varrho_2(\rho_a) \subset \tilde{U}_1; \\ \tilde{\pi}(x) &= \tilde{A} \circ \varrho_2((x_1, b)), \text{ if } x \in \tilde{A} \circ \varrho_2(\sigma_b) \subset \tilde{V}_1 \cup \tilde{W}_1. \end{aligned}$$

Therefore,

$$\tilde{\pi} \circ \phi_{\tilde{Z}} = \varphi_0 \circ \tilde{\pi} \quad \text{for } x \in \tilde{U}_1 \cup \tilde{V}_1 \cup \tilde{W}_1.$$

- (1) If $p \in \zeta^{-1}(\tilde{U}_1 \cup \tilde{V}_1 \cup \tilde{W}_1)$, define $\hat{\pi} = \zeta^{-1} \circ \tilde{\pi} \circ \zeta$, $\psi_1 = \hat{\psi}_1 = \zeta^{-1} \circ \phi_{\tilde{Z}} \circ \zeta$;
- (2) If $p \in \Omega \setminus \zeta^{-1}(\tilde{U}_1 \cup \tilde{V}_1 \cup \tilde{W}_1)$, define $\hat{\pi} = \text{id}$, $\psi_1 = \phi_{\hat{\alpha}_1 X}$, $\hat{\psi}_1 = \phi_{\alpha_1 X}$.

Proposition 3.12. ψ_1 and $\hat{\psi}_1$ are equivalent. Moreover,

$$h(\psi_1) > K_1, \quad h(\hat{\psi}_1) = 0.$$

Proof. The equivalence of ψ_1 and $\hat{\psi}_1$ are outputted by their constructions. It is left to estimate the entropies of ψ_1 and $\hat{\psi}_1$. Observing that $\phi_{\hat{\alpha}_1 X}$ is actually a factor of ψ_1 , that is, the following graph is commutative

$$\begin{array}{ccc} \Omega & \xrightarrow{\psi_1} & \Omega \\ \hat{\pi} \downarrow & & \downarrow \hat{\pi} \\ \Omega & \xrightarrow{\phi_{\hat{\alpha}_1 X}} & \Omega \end{array}$$

$$\hat{\pi} \circ \psi_1 = \phi_{\hat{\alpha}_1 X} \circ \hat{\pi}.$$

So by Theorem 7.2 of [46] and Proposition 3.10,

$$h(\psi_1) \geq h(\phi_{\hat{\alpha}_1 X}) > K_1.$$

Next we will show that $h(\hat{\psi}_1) = 0$. For every $p \in \Omega$ and $t > 0$, we need to estimate the proportion of its t -time orbit in \tilde{U} . Consider two transversal sections

$$H_1 = \{x = (x_1, x_2, \dots, x_{m+1}) \in B^{m+1}(0, 4) \mid x_1 = 3\}$$

$$H_2 = \{x = (x_1, x_2, \dots, x_{m+1}) \in B^{m+1}(0, 4) \mid x_1 = -3\}.$$

Given $x \in H_1$, denote by $\tau(\phi, x)$ the first time $t > 0$ satisfying that $\phi(x, t) \in H_2$. If $\phi(x, t) \notin H_2$ for all $t > 0$, we appoint $\tau(\phi, x) = \infty$. For $x \in H_1 \setminus \{x \mid \omega(x) \in F_0\}$ we claim that $\phi_{\tilde{Z}}(x, \tau(\phi_{\tilde{Z}}, x)) = \varphi_0(x, \tau(\varphi_0, x))$. To see why this is so, one can use the fact that

$$\tau(\phi_{\tilde{Z}}, x) = \tau(\varphi_0, x) \quad \text{and} \quad \tilde{\pi} \circ \phi_{\tilde{Z}} = \varphi_0 \circ \tilde{\pi}.$$

Exactly, $\tilde{\pi} = \text{id}$ for $x \in \Omega \setminus (\tilde{U}_1 \cup \tilde{V}_1 \cup \tilde{W}_1)$. Thus

$$\begin{aligned} \phi_{\tilde{Z}}(x, \tau(\phi_{\tilde{Z}}, x)) &= \tilde{\pi} \phi_{\tilde{Z}}(x, \tau(\phi_{\tilde{Z}}, x)) \\ &= \varphi_0(\tilde{\pi}(x), \tau(\varphi_0, \tilde{\pi}(x))) \\ &= \varphi_0(x, \tau(\varphi_0, x)). \end{aligned}$$

Recalling that for any open set $U_0 \subset \Omega \setminus (\tilde{U} \cup \tilde{V})$,

$$\lim_{t \rightarrow +\infty} \frac{J(t, p, \phi_{\alpha X}, U_0)}{t} = 0,$$

we know

$$\lim_{t \rightarrow +\infty} \frac{J(t, p, \widehat{\psi}_1, U_0)}{t} = 0,$$

which implies that $\widehat{\psi}_1$ has no invariant measures on $\Omega \setminus (\{p_0\} \cup \widetilde{U})$. Therefore

$$h(\widehat{\psi}_1) = 0.$$

□

Step 4 Smoothness of the flows ψ_1 and $\widehat{\psi}_1$.

Define a C^∞ function $v_0 : \mathbb{R}^2 \rightarrow \mathbb{R}$ as follows

$$v_0(x_1, x_2) = \begin{cases} e^{\frac{1}{\gamma_0^2(x_1) - x_2} + \frac{1}{x_2 - 4}} & \text{for } \gamma_0^2(x_1) < x_2 < 4, \\ 0 & \text{otherwise,} \end{cases}$$

which induces a new C^∞ function $\widehat{v}_0 : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by

$$\widehat{v}_0(x_1, x_2) = \frac{\int_{x_2}^4 v_0(x_1, s) ds}{\int_{\gamma_0^2(x_1)}^4 v_0(x_1, s) ds}.$$

We can verify that \widehat{v}_0 satisfies

- (1) $\widehat{v}_0(x_1, x_2) = 0$ for $|x_2| \geq 2$,
- (2) $\widehat{v}_0(x_1, x_2) = 1$ for $|x_2| \leq \gamma_0(x_1)$ and,
- (3) $\frac{\partial^{i+j} \widehat{v}_0}{\partial^i x_1 \partial^j x_2} |_{x_2=2} = \frac{\partial^{i+j} \widehat{v}_0}{\partial^i x_1 \partial^j x_2} |_{x_2=\gamma_0(x_1)} = 0$ for $i, j \geq 0$ and $i + j \geq 1$.

Using the function \widehat{v}_0 , we can define C^∞ vector field Z_1 for $(x_1, x_2) \in W_1$:

$$Z_1(x) = (\eta(x_1, x_2 - \widehat{v}_0(x_1, x_2)\gamma_0(x_1))(1, \widehat{v}_0(x_1, x_2)\gamma_0'(x_1)).$$

Let $Z_1(x) = Z(x)$ for $x \in U_1 \cup V_1$. Once more, we rotate Z_1 by $O(m)$ to obtain a vector field \widetilde{Z}_1 on $\widetilde{U}_1 \cup \widetilde{V}_1 \cup \widetilde{W}_1$. Then $\widetilde{Z}_1(x)$ is C^∞ on $\widetilde{V}_1 \cup \widetilde{W}_1$. Moreover, $\widetilde{Z}_1 = \widetilde{Z}$ is C^2 on $\widetilde{U}_1 \cup \widetilde{V}_1$. Consequently, \widetilde{Z}_1 is C^2 on $\widetilde{U}_1 \cup \widetilde{V}_1 \cup \widetilde{W}_1$.

Let $\widehat{\psi}_2 = \psi_2 = \zeta^{-1} \circ \phi_{\widetilde{Z}_1} \circ \zeta$, when $p \in \widetilde{U}$; $\psi_2 = \psi_1$, $\widehat{\psi}_2 = \widehat{\psi}_1$, when $p \in \Omega \setminus \widetilde{U}$.

Proposition 3.13. *ψ_2 and $\widehat{\psi}_2$ are equivalent. In addition $h(\widehat{\psi}_2) = 0$ and there is a constant $C_1 > 0$ independent of K_1 such that*

$$h(\psi_2) > C_1 K_1.$$

Proof. Noting the fact

$$\pi_i(\widetilde{Z}_1(x_1, x_2, \dots, x_{m+1})) = -\pi_i(\widetilde{Z}_1(-x_1, x_2, \dots, x_{m+1}))$$

for $2 \leq i \leq m+1$, we deduce

$$\begin{aligned}
& \pi_i(\phi_{\tilde{Z}_1}(x, \tau(x, \phi_{\tilde{Z}_1}))) \\
&= x + \int_0^{\tau(x, \phi_{\tilde{Z}_1})} \pi_i(\tilde{Z}_1(\phi_{\tilde{Z}_1}(x, s))) ds \\
&= x + \int_0^{\frac{\tau(x, \phi_{\tilde{Z}_1})}{2}} \pi_i(\tilde{Z}_1(\phi_{\tilde{Z}_1}(x, s))) ds + \int_{\frac{\tau(x, \phi_{\tilde{Z}_1})}{2}}^{\tau(x, \phi_{\tilde{Z}_1})} \pi_i(\tilde{Z}_1(\phi_{\tilde{Z}_1}(x, s))) ds \\
&= x + \int_0^{\frac{\tau(x, \phi_{\tilde{Z}_1})}{2}} \pi_i(\tilde{Z}_1(\phi_{\tilde{Z}_1}(x, s))) ds - \int_0^{\frac{\tau(x, \phi_{\tilde{Z}_1})}{2}} \pi_i(\tilde{Z}_1(\phi_{\tilde{Z}_1}(x, s))) ds \\
&= x
\end{aligned}$$

for $2 \leq i \leq m+1$, $x \in H_1 \setminus \{p \in \Omega \mid \omega(p) \in F_0\}$. Define $\tilde{\pi}_1 : B^{m+1}(0, 8) \rightarrow B^{m+1}(0, 8)$ as follows

$$\begin{aligned}
\tilde{\pi}_1(y) &= (\pi_1(\phi_{\tilde{Z}_1}(x, t)), x_2, \dots, x_{m+1}), \quad y = \phi_{\tilde{Z}_1}(x, t), \quad x \in H_1, \quad 0 \leq t < \tau(\varphi_2, x); \\
\tilde{\pi}_1(y) &= y, \quad \text{otherwise.}
\end{aligned}$$

And further define $\hat{\pi}_1 : \Omega \rightarrow \Omega$ by

$$\hat{\pi}_1 = \begin{cases} \zeta^{-1} \circ \tilde{\pi}_1 \circ \zeta, & \text{if } p \in \tilde{U}; \\ \text{id}, & \text{if } p \in \Omega \setminus \tilde{U}. \end{cases}$$

Then ψ_2 and $\hat{\psi}_2$ are equivalent given by $\hat{\pi}_1$.

Observing that there is no singularity in W_1 , we can choose $0 < C_1 < 1, C_2 > 1$ such that

$$C_1 < \frac{\tau(\phi_{\tilde{Z}_1}, x)}{\tau(\phi_{\tilde{Z}}, x)} < C_2 \quad \text{for } x \in H_1 \setminus \{p \mid \omega(p) \in A_0\}.$$

By Proposition 3.12 and the variational principle, there exists an ergodic measure μ_1 of ψ_1 such that

$$h_{\mu_1}(\psi_1) > K_1.$$

Obviously,

$$\text{supp}(\mu_1) \cap \zeta^{-1}(\tilde{U}_1) = \emptyset.$$

Let $\nu_1 = \mu_1 \mid \zeta^{-1}(H_1)$. Given a flow ϕ on Ω , for any $p \in \zeta^{-1}H_1$, denote by $T(\phi, p) > 0$ the first time of p returning $\zeta^{-1}(H_1)$, and let the return map

$$R(\phi, p) = \phi(p, T(\phi, p)) \in \zeta^{-1}(H_1).$$

By Abarmov Theorem [3],

$$\frac{h_{\nu_1}(R(\psi_1))}{\int_{\zeta^{-1}(H_1)} T(\psi_1, p) d\nu_1} = h_{\mu_1}(\psi_1) > K_1.$$

For $t > 0$ large, define two sequences Γ_i, Γ'_i of sub-orbit of $\{\phi(p, s) \mid 0 \leq s \leq t\}$ as follows. We begin with p . Let Γ_i be the sequence of minimal intervals whose left endpoint lies in $\zeta^{-1}(H_1)$ and right endpoint lies in $\zeta^{-1}(H_2)$. Let Γ'_i be the sequence of minimal intervals whose left endpoint lies in $\zeta^{-1}(H_2)$ and right endpoint lies in $\zeta^{-1}(H_1)$.

For each interval Γ , let $|\Gamma|$ denote the time of sub-orbit Γ .

Since $\psi_2 = \psi_1$ for $p \notin \tilde{U}$, we have

$$\begin{aligned} \frac{T(\psi_2, p)}{T(\psi_1, p)} &= \frac{\sum \Gamma_i(\psi_2) + \sum \Gamma'_i(\psi_2)}{\sum \Gamma_i(\psi_1) + \sum \Gamma'_i(\psi_1)} \\ &\leq \frac{\sum \Gamma_i(\psi_2) + C_2 \sum \Gamma'_i(\psi_1)}{\sum \Gamma_i(\psi_1) + \sum \Gamma'_i(\psi_1)} \\ &\leq C_2, \end{aligned}$$

and on the other hand,

$$\frac{\sum \Gamma_i(\psi_2) + \sum \Gamma'_i(\psi_2)}{\sum \Gamma_i(\psi_1) + \sum \Gamma'_i(\psi_1)} \geq \frac{C_1 \sum \Gamma_i(\psi_2) + \sum \Gamma'_i(\psi_2)}{\sum \Gamma_i(\psi_1) + \sum \Gamma'_i(\psi_1)} \geq C_1.$$

So,

$$C_1 < \frac{T(\psi_2, p)}{T(\psi_1, p)} < C_2 \quad \text{for } p \in \zeta^{-1}(H_1) \setminus \{p \in \Omega \mid \omega(p) \in F_0\}.$$

We can define a measure μ_2 by

$$\int_{\Omega} g d\mu_2 := \int_{\zeta^{-1}(H_1)} \int_0^{T(\psi_2, p)} g(\psi_2(p, t)) dt d\nu_1, \quad \forall g \in C^0(\Omega).$$

Using Lemma 3.5, μ_2 is an ergodic invariant measure of ψ_2 . Furthermore, $R(\psi_1) = R(\psi_2)$ together with Abramov Theorem [3] yields that

$$\begin{aligned} h_{\mu_2}(\psi_2) &= \frac{h_{\nu_1}(R(\psi_2))}{\int_{\zeta^{-1}(H_1)} T(\psi_2, p) d\nu_1} \\ &= \frac{h_{\nu_1}(R(\psi_1))}{\int_{\zeta^{-1}(H_1)} T(\psi_1, p) d\nu_1} \frac{\int_{\zeta^{-1}(H_1)} T(\psi_1, p) d\nu_1}{\int_{\zeta^{-1}(H_1)} T(\psi_2, p) d\nu_1} \\ &\geq C_1 h_{\mu_1}(\psi_1) \\ &> C_1 K_1. \end{aligned}$$

Therefore

$$h(\psi_2) \geq h_{\mu_2}(\psi_2) > C_1 K_1.$$

Finally, since all invariant measures of $\hat{\psi}_2$ are supported on singularities so $h(\hat{\psi}_2) = 0$. \square

Step 5 Embed the two dimensional flows ϕ_{Z_1} and ϕ_{Z_2} into Ω .

At most taking a scallion of the coordinate (\tilde{U}, ζ) , we assume that $B^{m+1}(0, 3) \subset \zeta(\tilde{U}_1)$. In this subsection, all modifications will be completed in $B^{m+1}(0, 3)$. Denote

$$\begin{aligned} D_1 &= \{x \in \mathbb{R}^{m+1} \mid x_1^2 + x_2^2 \leq 1 \text{ and } \sum_{i=3}^{m+1} x_i^2 = 0\}, \\ D_2 &= \{x \in \mathbb{R}^{m+1} \mid \sum_{i=1}^{m+1} x_i^2 < 2\}. \end{aligned}$$

We first choose C^∞ smooth functions $\varsigma, \widehat{z}_1, \widehat{z}_2, \widehat{\beta}_1, \widehat{\beta}_2(x) : \widetilde{U}_1 \rightarrow \mathbb{R}$ such that

$$\varsigma(x) \begin{cases} = 0 & x \in D_1, \\ > 0 & x \in D_2 \setminus D_1, \\ = 0 & x \in \widetilde{U}_1 \setminus D_2; \end{cases}$$

$$\widehat{z}_1(x) = \begin{cases} -x_2 + \alpha(x_1^2 + x_2^2)x_1 & x \in D_1, \\ 0 & x \in \widetilde{U}_1 \setminus D_2; \end{cases}$$

$$\widehat{z}_2(x) = \begin{cases} -x_1 + \alpha(x_1^2 + x_2^2)x_2 & x \in D_1, \\ 0 & x \in \widetilde{U}_1 \setminus D_2; \end{cases}$$

$$\widehat{\beta}_1(x) = \begin{cases} \beta_1(x) & x \in D_1, \\ \widehat{\beta}_1(x) > 0 & x \in D_2 \setminus D_1 \\ 1 & x \in \widetilde{U}_1 \setminus D_2; \end{cases}$$

$$\widehat{\beta}_2(x) = \begin{cases} \beta_2(x) & x \in D_1, \\ \widehat{\beta}_2(x) > 0 & x \in D_2 \setminus D_1 \\ = 1 & x \in \widetilde{U}_1 \setminus D_2. \end{cases}$$

Let

$$\begin{aligned} \widehat{Z}_1(x) &= \widehat{\beta}_1(x)(\widehat{z}_1(x), \widehat{z}_2(x), \varsigma(x), \dots, \varsigma(x)), \\ \widehat{Z}_2(x) &= \widehat{\beta}_2(x)(\widehat{z}_1(x), \widehat{z}_2(x), \varsigma(x), \dots, \varsigma(x)). \end{aligned}$$

Noting that $\varsigma(x) > 0$ for $x \in D_2 \setminus D_1$, we know that there is no nonwandering points in $D_2 \setminus D_1$ for both $\phi_{\widehat{Z}_1}$ and $\phi_{\widehat{Z}_2}$. Hence, all invariant measures on D_2 are supported on periodic orbits in D_1 , which implies no entropy production in \widetilde{U}_1 . Define

$$\widehat{X}_1(p) = \begin{cases} (d\zeta^{-1})\widehat{Z}_1(\zeta(p)) & p \in \zeta^{-1}(\widetilde{U}_1), \\ \frac{d\psi_2(p,t)}{dt} \big|_{t=0} & p \in \Omega \setminus \zeta^{-1}(\widetilde{U}_1); \end{cases}$$

$$\widehat{X}_2(p) = \begin{cases} (d\zeta^{-1})\widehat{Z}_2(\zeta(p)) & p \in \zeta^{-1}(\widetilde{U}_1), \\ \frac{d\widehat{\psi}_2(p,t)}{dt} \big|_{t=0} & p \in \Omega \setminus \zeta^{-1}(\widetilde{U}_1). \end{cases}$$

Then $\zeta \circ \phi_{\widehat{X}_1} \circ \zeta^{-1} \big|_{D_1} = \phi_{Z_1}$, $\zeta \circ \phi_{\widehat{X}_2} \circ \zeta^{-1} \big|_{D_1} = \phi_{Z_2}$. By Theorem A it holds that

$$EP(\phi_{\widehat{X}_1}) = \infty \text{ and } EP(\phi_{\widehat{X}_2}) = 0.$$

Finally, let $\psi = \phi_{\widehat{Z}_2}$, $\widehat{\psi} = \phi_{\widehat{Z}_1}$ and take $K_1 C_1 > K$. We conclude that

$$EP(\psi) = 0 \text{ and } EP(\widehat{\psi}) = \infty,$$

$$h(\psi) > K \text{ and } h(\widehat{\psi}) = 0.$$

4. FINAL REMARKS

While our results give a very complete answer to the degeneration of the growth of periodic orbits for two-dimensional equivalent flows in the category of C^∞ some interesting problems remain, that we pose here

Question 4.1. Is it possible to construct an analytic vector field or analytic map with $EP = \infty$? Our method of proof clearly cannot be made analytic since α_0 is flat at 0. Noting that for any k -order polynomial map P_k on \mathbb{R}^l , any n periodic point x of P_k satisfies

$$P_k^n(x) - x = 0.$$

By the Bezout theorem the number of isolated solutions is at most k^{nl} , which implies

$$EP(P_k) \leq \limsup_{n \rightarrow +\infty} \frac{1}{n} \log(k^{nl}) = l \log k < \infty.$$

This fact make us tend to consider the answer to be negative.

Question 4.2. Is the extreme $EP = 0$ or $EP = \infty$ or the sign of EP with finite value preserved by orbit equivalent analytic flows? The flows ψ and $\hat{\psi}$ are not analytic since ω_1 and $\hat{\omega}_1$ are flat at p_0 .

Question 4.3. Is the extreme $EP = 0$ or $EP = \infty$ or the sign of EP with finite value preserved by equivalent differential flows with only hyperbolic orbits? Recall that a periodic orbit $\{\phi(x, t) \mid 0 \leq t \leq T\}$ of period T is called hyperbolic if the linearization $D\phi(\cdot, T)$ at x has no eigenvalue in the unity circle except the flow direction.

We also have questions concerning entropy h and its relation to EP in smooth regularity.

Question 4.4. Is the value zero or the sign of entropy preserved by equivalent analytic flows?

Furthermore

Question 4.5. Are there two equivalent C^∞ or even analytic flows, one of which has positive topological entropy and zero exponential growth rate of periodic orbits, in contrast, the other has zero topological entropy and super-exponential growth of periodic orbits? In our constructions, η is only C^2 and can't be improved due to the appearance of square root when we use rotations of vector fields on $U_1 \cup V_1$.

Question 4.6. Besides entropy and the exponential growth of periodic orbits, are there other objects invariant or decreasing for equivalent flows? Actually physical measures [37] and Lyapunov exponents [19] could decrease for equivalent flows.

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